Expanding on Homological Algebra

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Recollections

In homological algebra one studies chain complexes, i.e. modules C_n for $n \geq 0$ together with homomorphisms $\partial_n : C_n \longrightarrow C_{n-1}$ such that $\partial_n \partial_{n+1} = 0$. Diagramatically we have

$$C : \qquad \cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} C_0 \longrightarrow C_0$$

Let us call the above chain complex C and let

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- **Simplex** The n-simplex is the $n+1$-tuple $[0, 1, 2 \ldots n]$. The components are called the **vertices** of the simplex.
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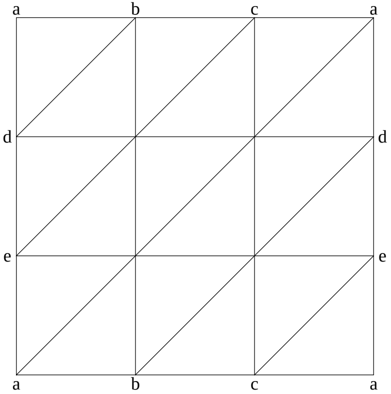
Simplicial Homology

Let X be a triangulated space. This means that X is constructed by gluing together simpleces.

Example



is a representation of the torus.

Simplicial Homology – cont.

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